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An inequality for characteristic functions and its applications to uncertainty relations and the quantum Zeno effect

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Abstract
An inequality concerning characteristic functions is established. It is useful in studying zero neighbourhood behaviours of characteristic functions. The physical implications for the time–energy uncertainty relations and the quantum Zeno effect are indicated.

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Let F be a probability distribution function on $\mathbb{R}$, that is, F is non-decreasing, left continuous, $F(-\infty) = 0$, $F(\infty) = 1$. Let

$$\Phi(t) := \int \mathrm{e}^{-\mathrm{i}tx}\,\mathrm{d}F(x)$$

be its characteristic function (Fourier transform). It is well known that if F has finite second moment M_2, then

$$|\Phi(t)| \geqslant 1 - \tfrac{1}{2}M_2|t|^2 \qquad \forall t \in \mathbb{R}.$$

Motivated by several questions such as parameter-based time–energy uncertainty relations and the quantum Zeno effect in quantum mechanics, we are led to consider the following problem: for any $0 \leqslant \beta \leqslant 2$, does there exist a positive constant α, depending only on β and F, such that

$$|\Phi(t)| \geqslant 1 - \alpha|t|^{\beta} \qquad \forall t \in \mathbb{R}?$$

A solution of this problem will have interesting applications in both statistics and physics. It has applications in estimating the first zero of characteristic functions, in studying the decay rate of unstable systems, in quantifying the Heisenberg uncertainty principle [6, 8, 10, 11], in providing a simple criterion for the occurrence of the quantum Zeno effect [1, 2, 7], etc.

To the best of our knowledge, this question has not been addressed before. This paper presents a solution of the above problem which is summarized in theorem 1. By virtue of this result, we obtain a family of parameter-based time–energy uncertainty relations (corollary 1), and a simple sufficient condition for the occurrence of the quantum Zeno effect (corollary 2).

Theorem 1. (i) *For any* $0 \leqslant \beta \leqslant 2$*, it holds that*

$$|\Phi(t)| \geqslant 1 - \lambda_\beta M_\beta |t|^\beta \qquad \forall t \in \mathbb{R}. \tag{1}$$

Here

$$M_\beta := \int |x|^\beta \, \mathrm{d}F(x)$$

is the β*th absolute moment of* F *and* λ_β *is defined as*

$$\lambda_\beta := \sup_{x \geqslant 0} \frac{1 - \cos x}{x^\beta}$$

which satisfies $\lambda_0 = 2$, $\lambda_2 = 1/2$ *and*

$$1 - \cos 1 \leqslant \lambda_\beta \leqslant 2^{1-\beta} \qquad \forall \beta \in [0, 2]. \tag{2}$$

(ii) *For any fixed* $\beta > 2$*, if* F *is non-degenerate (that is,* F *is not concentrated on one point), and* $M_\beta < \infty$*, then there does not exist any constant* $\alpha > 0$ *(depends only on* β *and* F*) such that*

$$|\Phi(t)| \geqslant 1 - \alpha |t|^\beta \qquad \forall t \in \mathbb{R}. \tag{3}$$

Proof. (i) If $M_\beta = \infty$, the result is trivially true, thus we only need to prove inequality (1) when $M_\beta < \infty$. Since $|\Phi(t)| \geqslant \mathrm{Re}\,\Phi(t)$ (the real part of $\Phi(t)$), it suffices to prove

$$1 - \mathrm{Re}\,\Phi(t) \leqslant \lambda_\beta M_\beta |t|^\beta.$$

Since for $0 \leqslant \beta \leqslant 2$, $t > 0$ (for $t = 0$, inequality (1) is trivially true since $\Phi(0) = 1$)

$$\frac{1 - \cos(tx)}{|tx|^\beta} \leqslant \sup_{x \geqslant 0} \frac{1 - \cos(tx)}{|tx|^\beta} = \sup_{x \geqslant 0} \frac{1 - \cos x}{x^\beta} = \lambda_\beta$$

by noting $\cos(-tx) = \cos(tx)$, we have

$$\begin{aligned} 1 - \mathrm{Re}\,\Phi(t) &= \int (1 - \cos(tx)) \, \mathrm{d}F(x) \\ &\leqslant \int \lambda_\beta |tx|^\beta \, \mathrm{d}F(x) \\ &= \lambda_\beta M_\beta |t|^\beta. \end{aligned}$$

We now prove $\lambda_\beta \leqslant 2^{1-\beta}$. First note that if $0 \leqslant \gamma \leqslant 1$, then

$$|\sin x| \leqslant |x|^\gamma \qquad \forall x \in \mathbb{R}. \tag{4}$$

In fact, when $|x| \geqslant 1$, we have $|\sin x| \leqslant 1 \leqslant |x|^\gamma$ for any $\gamma \geqslant 0$; when $|x| \leqslant 1$, we have $|\sin x| \leqslant |x| \leqslant |x|^\gamma$ for any $0 \leqslant \gamma \leqslant 1$. Consequently, when $0 \leqslant \gamma \leqslant 1$, inequality (4) always holds.

Now since $0 \leqslant \beta/2 \leqslant 1$, by inequality (4), we have

$$1 - \cos(tx) = 2\sin^2(tx/2) \leqslant 2\left(|tx/2|^{\beta/2}\right)^2 = 2^{1-\beta}|tx|^\beta$$

and the inequality $\lambda_\beta \leqslant 2^{1-\beta}$ follows.

Next, we prove that $\min_{\beta\in[0,2]} \lambda_\beta = 1 - \cos 1$. Since $\lambda_0 = 2$ and $\lambda_2 = \frac{1}{2}$, we only need to consider $\beta \in (0, 2)$. Fixing such a β, and taking the partial derivative of the function

$$\Lambda_\beta(x) := \frac{1 - \cos x}{x^\beta} \qquad x \geqslant 0$$

with respect to x, we have (the case $\beta = 1$ is interpreted in a limiting sense)

$$\frac{\partial \Lambda_\beta(x)}{\partial x} = \frac{\sin x \cdot x^\beta - (1 - \cos x)\beta x^{\beta-1}}{x^{2\beta}} = \frac{\sin x}{x^{\beta+1}}(x - \beta \tan(x/2)).$$

A more detailed analysis shows that for fixed $\beta \in (0, 2)$, $\Lambda_\beta(x)$ attains its maximum value when x is the first positive root of the equation $x - \beta \tan(x/2) = 0$. We denote this solution as $x(\beta)$ to emphasize its dependence on β. Therefore

$$\lambda_\beta := \sup_{x\geqslant 0} \frac{1 - \cos x}{x^\beta} = \frac{1 - \cos(x(\beta))}{x^\beta(\beta)} = \Lambda_\beta(x(\beta)).$$

Now taking the derivative of the above function with respect to β, and noting that $\frac{\partial}{\partial x}\Lambda_\beta(x)\big|_{x=x(\beta)} = 0$, we obtain

$$\frac{\partial \lambda_\beta}{\partial \beta} = \left(\frac{\partial \Lambda_\beta(x)}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial \Lambda_\beta(x)}{\partial \beta}\right)\Big|_{x=x(\beta)} = -\ln x \frac{1 - \cos x}{x^\beta}\Big|_{x=x(\beta)}.$$

Therefore, whenever $x(\beta) > 1$, λ_β as a function of β is decreasing, and whenever $x(\beta) < 1$, λ_β as a function of β is increasing. But the first positive solution $x = x(\beta)$ of the equation $x - \beta \tan(x/2) = 0$ is a decreasing function of β. We have $x(\beta) > 1$ when $\beta < 1/\tan(1/2)$, and $x(\beta) < 1$ when $\beta > 1/\tan(1/2)$. Consequently, when $\beta = 1/\tan(1/2)$ (in this case $x(\beta) = 1$), λ_β takes its minimum value which is equal to

$$\Lambda_\beta(x(\beta))|_{\beta=1/\tan(1/2)} = 1 - \cos 1 \approx 0.4597.$$

(ii) If F is non-degenerate and $M_\beta < \infty$ for some $\beta > 2$, then we have $0 < M_2 < \infty$. By Taylor expansion,

$$\Phi(t) = 1 - \mathrm{i}bt - \tfrac{1}{2}M_2 t^2 + o(t^2) \qquad \text{for small } |t| \in \mathbb{R}.$$

Here $b := \int x \, \mathrm{d}F(x)$ is the expectation of F. From this we obtain

$$|\Phi(t)|^2 = 1 - (M_2 - b^2)t^2 + o(t^2) \qquad \text{for small } |t| \in \mathbb{R}.$$

Consequently,

$$|\Phi(t)| = 1 - \tfrac{1}{2}\sigma^2 t^2 + o(t^2) \qquad \text{for small } |t| \in \mathbb{R}.$$

Here $\sigma^2 := M_2 - b^2$ is the variance of F. But since F is non-degenerate, the variance σ^2 is never zero. Now if inequality (3) also holds true, then by noting $\beta > 2$, we have

$$1 = \lim_{t\to 0} \frac{1 - |\Phi(t)|}{1 - |\Phi(t)|} \leqslant \lim_{t\to 0} \frac{\alpha|t|^\beta}{\sigma^2 t^2/2 - o(t^2)} = 0.$$

A contradiction! □

Remark.

(1) It is obvious that if $\beta > 2$, then

$$\lambda_\beta := \sup_{x\geqslant 0} \frac{1 - \cos x}{x^\beta} = \infty$$

since $1 - \cos x = \frac{1}{2}x^2 + o(x^2)$ when x is near zero. The graph of λ_β as a function of β is plotted in figure 1.

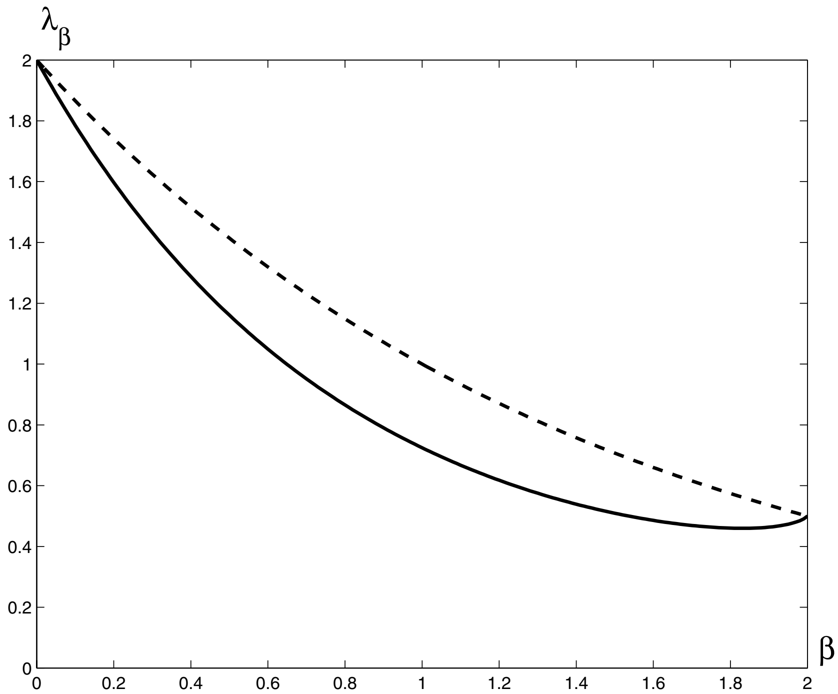

Figure 1. The solid curve is the graph of $\lambda_\beta = \sup_{x \geqslant 0} \frac{1-\cos x}{x^\beta}$ and the dashed curve is the graph of $\lambda_\beta = 2^{1-\beta}$. The minimum of λ_β when β changes on [0, 2] occurs when $\beta = 1/\tan(1/2) \approx 1.8305$, and $\min_{\beta\in[0,2]} \lambda_\beta = 1 - \cos 1 \approx 0.4597$.

(2) Inequality (1) for $\beta = 1$ has been used to prove a variation of the Heisenberg uncertainty relations involving an average [4].

Corollary 1. *Let* $\tau := \inf\{t \geqslant 0 : \Phi(t) = 0\}$ *be the first zero of the characteristic function* Φ. *If* $0 \leqslant \beta \leqslant 2$, *then*

$$\tau \geqslant \frac{1}{(\lambda_\beta M_\beta)^{1/\beta}}. \tag{5}$$

Proof. Readily follows from inequality (1) and $\Phi(\tau) = 0$. □

Corollary 2. *If there exists* $\beta > 1$ *such that* $M_\beta < \infty$, *then for any fixed* $t \in \mathbb{R}$,

$$\lim_{n\to\infty} |\Phi(t/n)|^n = 1. \tag{6}$$

Proof. Since Φ is the characteristic function of a probability distribution, we have $|\Phi(t)| \leqslant 1$, thus $\lim_{n\to\infty} |\Phi(t/n)|^n \leqslant 1$. On the other hand, from inequality (1), we have

$$\begin{aligned}\lim_{n\to\infty} |\Phi(t/n)|^n &\geqslant \lim_{n\to\infty} (1 - \lambda_\beta M_\beta |t/n|^\beta)^n \\ &= \lim_{n\to\infty} \exp\{-\lambda_\beta M_\beta |t|^\beta n^{1-\beta}\} \\ &= 1.\end{aligned}$$
□

Remark. The condition that $M_\beta < \infty$ is a sufficient, but not a necessary, condition for (6) to hold. We conjecture that (6) holds true if and only if $M_1 < \infty$.

Now we discuss the physical interpretations of our results. Let H be a Hamiltonian and $|\psi\rangle$ be a quantum state. We want to estimate the evolution speed of $|\psi\rangle$ under H by virtue of moments. Let

$$\tau = \inf\{t \geqslant 0 : \langle\psi|\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle = 0\}$$

be the first time that $|\psi\rangle$ evolves into an orthogonal state (we put the Planck constant $\hbar = 1$). Two fundamental results along this line are $\tau\Delta H \geqslant \frac{\pi}{2}$, which gives an estimate of τ in terms of the variance [5, 10, 11], and $\tau\langle H\rangle \geqslant \frac{\pi}{2}$ (for non-negative H), which gives an estimate of τ in terms of the average [6]. However, when the variance and the average of H are infinite, these two results only give rise to the trivial estimation: $\tau \geqslant 0$.

Let $\{|E\rangle\}$ be the complete set of energy eigenstates: $H|E\rangle = E|E\rangle$, $\langle E'|E\rangle = \delta(E - E')$. Let $|\psi\rangle$ and $|\mathrm{e}^{-\mathrm{i}tH}\psi\rangle$ be expanded in the energy eigenstates (that is, written as superpositions of the complete set $\{|E\rangle\}$) as

$$|\psi\rangle = \int \lambda(E)|E\rangle\,\mathrm{d}E$$

$$|\mathrm{e}^{-\mathrm{i}tH}\psi\rangle = \mathrm{e}^{-\mathrm{i}tH}\int \lambda(E)|E\rangle\,\mathrm{d}E = \int \lambda(E)\,\mathrm{e}^{-\mathrm{i}tE}|E\rangle\,\mathrm{d}E.$$

Thus $\lambda(E) = \langle E|\psi\rangle$ is the wavefunction of the quantum state in the energy representation. When the energy spectrum is discrete, the integrals should be interpreted as discrete sums. Let $A(t) := \langle\psi|\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle$ be the survival amplitude, then

$$A(t) = \langle\psi|\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle = \int |\lambda(E)|^2\,\mathrm{e}^{-\mathrm{i}tE}\,\mathrm{d}E.$$

Consequently, the survival amplitude $A(t) = \langle\psi|\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle$ is precisely the characteristic function (Fourier transform) of the state energy distribution $|\lambda(E)|^2 = |\langle E|\psi\rangle|^2$. Identifying $A(t)$ with $\Phi(t)$, $|\lambda(x)|^2\,\mathrm{d}x$ with $\mathrm{d}F(x)$, E with x and t with time in theorem 1 and corollary 1, then τ is the first positive zero of Φ (physically the first time that the quantum state evolves into an orthogonal state). Thus inequality (5) may be viewed as a general parameter-based time–energy uncertainty relation: the evolution speed (here characterized by τ) is controlled by the energy spread (here characterized by M_β). Furthermore, the limiting property (6) is precisely the mathematical expression for the occurrence of the quantum Zeno effect, the phenomenon that frequent observations (measurements) will slow down or even inhibit the evolution of the decay of an unstable system [1, 2, 7]. Therefore, corollary 2 provides a simple sufficient condition for the occurrence of the quantum Zeno effect: whenever there exists a $\beta > 1$ such that $M_\beta < \infty$, then $\lim_{n\to\infty}|A(t/n)|^{2n} = 1$, consequently, the quantum Zeno effect exhibits manifestation. This result improves a somewhat controversial issue concerning the quantum Zeno effect [1, 2]: in the literature, one usually requires $M_2 < \infty$ to guarantee the quantum Zeno effect. The condition is also very close to being necessary in the sense that if $M_1 = \infty$, then as clearly seen from the subsequent example, $\lim_{n\to\infty}|A(t/n)|^{2n}$ may be strictly less than 1 even if the limit exists. It seems that the necessary and sufficient condition for the occurrence of the quantum Zeno effect is $M_1 < \infty$. The following discussion supports this claim.

Finally, let us see an illustrative example. Let ν be a positive constant. Consider the state energy distribution

$$|\lambda(E)|^2 = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)}(1 + E^2)^{-\nu-\frac{1}{2}} \qquad E \in \mathbb{R}. \tag{7}$$

In statistical terms, this distribution is Pearson's seventh distribution, when $\alpha = \frac{1}{2}$, it is the celebrated Breit–Wigner distribution (Lorentzian distribution or Cauchy distribution). This distribution has a finite absolute βth moment, that is, $M_\beta < \infty$, if and only if $\beta < 2\nu$. Its characteristic function is (see [9], p 254)

$$A(t) := \int \mathrm{e}^{-\mathrm{i}tE}|\lambda(E)|^2\,\mathrm{d}E = \frac{2}{\Gamma(\nu)}\left(\frac{|t|}{2}\right)^\nu K_\nu(|t|) \qquad t \in \mathbb{R}.$$

Here $K_\nu(z)$ is the modified Bessel function defined as

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \qquad -\pi < \arg z < \pi$$

and for any $\nu \in \mathbb{R}$,

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu+1)n!}\left(\frac{z}{2}\right)^{2n} \qquad z \in \mathbb{C}.$$

We assume that ν is not an integer. When ν is an integer, the above formulae still hold if interpreted in the limiting sense. After some simple manipulations and noting that $\Gamma(\nu)\Gamma(-\nu+1) = \frac{\pi}{\sin(\nu\pi)}$, we readily see that when $t > 0$ is near zero,

$$A(t) = 1 - \frac{\pi}{\Gamma(\nu)\Gamma(\nu+1)\sin(\nu\pi)}\left(\frac{t}{2}\right)^{2\nu} + O(t^2).$$

Consequently,

$$|A(t/n)|^{2n} = \left(1 - \frac{\pi}{\Gamma(\nu)\Gamma(\nu+1)\sin(\nu\pi)}\left(\frac{t}{2}\right)^{2\nu} + O(t^2)\right)^{2n}$$

$$\approx \exp\left\{-\frac{\pi t^{2\nu}}{2^{2\nu-1}\Gamma(\nu)\Gamma(\nu+1)\sin(\nu\pi)}n^{1-2\nu} + O(t^2/n)\right\} \qquad \text{(when } n \text{ is large)}$$

from which we get

$$\lim_{n\to\infty} |A(t/n)|^{2n} = \begin{cases} 0 & \nu < 1/2 \\ \mathrm{e}^{-t} & \nu = 1/2 \\ 1 & \nu > 1/2. \end{cases}$$

Consequently, there are three distinct scenarios:

(1) When $\nu = \frac{1}{2}$, the decay is exactly exponential. In this case, we note that $M_1 = \infty$.

(2) When $\nu > \frac{1}{2}$, the decay slows down for small *t*, and the quantum Zeno effect manifests. Note that in this case, $M_\beta < \infty$ for $\beta < 2\nu$. Since $\nu > \frac{1}{2}$, we can always take a β satisfying $1 < \beta$ and $M_\beta < \infty$, thus in conformity with corollary 2.

(3) When $\nu < \frac{1}{2}$, the decay is actually accelerated (compared with the exponential decay); we cannot obtain the quantum Zeno effect, in contrast, the survival probability tends to zero, and we get the quantum anti-Zeno effect [3].

Furthermore, when $\nu \leqslant \frac{1}{2}$, the average of the distribution of $|\lambda(E)|^2$ is infinite, and the classical inequalities $\tau\Delta H \geqslant \frac{\pi}{2}$ and $\tau\langle H\rangle \geqslant \frac{\pi}{2}$ give only the trivial estimate: $\tau \geqslant 0$ (since $\Delta H = \infty$, $\langle H\rangle = \infty$). However, corollary 1 still gives a non-trivial estimate of τ if we take any $\beta < 2\nu$:

$$\tau \geqslant \left(\lambda_\beta \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\nu)} \mathrm{B}\left(\frac{\beta}{2}+\frac{1}{2}, \nu - \frac{\beta}{2}\right)\right)^{-1/\beta}.$$

Here $\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x \equiv \int_0^\infty x^{p-1}(1+x)^{-p-q}\,\mathrm{d}x$ is the beta integral ($p > 0$, $q > 0$). Actually $\tau = \infty$ in this example.

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